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Solving Discretely-Constrained Nash–Cournot Games with an Application to Power Markets

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Abstract This paper provides a methodology to solve Nash–Cournot energy production games allowing some variables to be discrete. Normally, these games can be stated as mixed complementarity problems but only permit continuous variables in order to make use of each producer's Karush–Kuhn–Tucker conditions. The proposed approach allows for more realistic modeling and a compromise between integrality and complementarity to avoid infeasible situations.

Keywords Nash · Cournot · Integer · Discrete · Game theory · Power market

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1 Introduction

This paper considers a Nash–Cournot game between energy producers in which each player solves a discretely-constrained optimization problem. Typically, such an optimization problem maximizes producer profits subject to operational and investment decisions. By taking the Karush–Kuhn–Tucker (KKT; Bazaraa et al. 1993) conditions to each player's problem and combining them, perhaps with additional market-clearing conditions in some cases, results in a mixed complementarity problem (MCP; Cottle et al. 1992) which has recently seen a lot of applications in energy and other issues (e.g., Bard 1983, 1988; Bard and Moore 1990; Karlof and Wang 1996; Labbé et al. 1998; Luo et al. 1996; Moore and Bard 1990; Wen and Huang 1996) and more recently (e.g., Bard et al. 2000, Fuller 2008, 2010 (personal communication); Gabriel and Leuthold 2010; Gabriel et al. 2010; Hu et al. 2009; Marcotte et al. 2001; O'Neill et al. 2005; Scaparra and Church 2008). Equilibrium problems, in general, have been well formulated and studied in power markets (García-Bertrand et al. 2005; Leuthold et al. 2012; Metzler et al. 2003; Oggioni et al. 2011; Smeers 2003) as well as natural gas markets (Abada et al. 2012; Siddiqui and Gabriel 2012).

Crucial to expressing the Nash–Cournot game between two or more energy producers as an MCP is the assumption that the KKT conditions can be formulated. When for example some of the variables are integer-valued (e.g., binary go/no go decisions), the KKT conditions are not valid. In this paper we show a new approach that provides a compromise between complementarity and integrality. This is done by first relaxing the discretely-constrained variables to their continuous analogs, taking KKT conditions for this relaxed problem, converting these conditions to disjunctive-constraints form (Fortuny-Amat and McCarl 1981), and then solving them along with the original integer restrictions re-inserted in a mixed-integer, linear program (MILP). The integer conditions are then further relaxed, but targeted using penalty terms in the objective function. This MILP relaxes both complementarity and integrality but tries to find minimum deviations for both and as such is an example of bi-objective problem (Cohon 1978). When an equilibrium solution exists that additionally satisfies the integrality conditions, we show that it can be found.

In Section 2 we first provide the general form of this problem which we call discretely-constrained mixed linear complementarity problem (DC-MLCP) and which was originally stated in Gabriel et al. (2012, 2013) but not specialized to the current context. The MILP that is used to solve the DC-MLCP is shown to have a solution under mild conditions. Then, we specialize the DC-MLCP to the current context of a discretely-constrained Nash–Cournot game between energy producers.

Nash–Cournot problems without discrete variables have long been studied and it is well known that they can be expressed either as nonlinear complementarity or variational inequality problems (Facchinei and Pang 2003). Allowing for discrete variables makes the problem more realistic as in some settings, for example, production can only occur in discrete amounts or there are binary decisions about starting up/shutting down or investing in some generation or transmission capacity. We show a correspondence between the solution set to the discretely-constrained Nash game and integer solutions to the continuous relaxation.

Section 2 provides the mathematical formulation of the considered model and describes the proposed solution technique. Section 3 provides numerical examples that validate the proposed approach followed by conclusions and extensions in Section 4 and an Appendix with specific key formulations.

2 Problem definition

Consider a general, discretely-constrained mixed linear complementarity problem. The formulation is as follows: given the vector $q = \begin{pmatrix} q_1 \ q_2 \end{pmatrix}^T$ and matrix $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, find $z = \left(z_1^T, z_2^T\right)^T \in R^{n_1} \times R^{n_2}$ such that:

$$0 \leq q_1 + \begin{pmatrix} A_{11} & A_{12} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \perp z_1 \geq 0 \tag{1a}$$

$$0 = q_2 + \begin{pmatrix} A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, z_2 \text{ free} \tag{1b}$$

$$(z_1)_c \in R_+,\ c \in C_1, (z_1)_d \in Z_+, d \in D_1 \tag{1c}$$

$$(z_2)_c \in R,\ c \in C_2, (z_2)_d \in Z, d \in D_2 \tag{1d}$$

We partition the indices for $z_i, i = 1, 2$ into continuous-valued (denoted by the set C_i) and discrete-valued variables (denoted by the set D_i), i.e., $z_i = \left((z_i)_{C_i}^T \ (z_i)_{D_i}^T \right)^T, i = 1, 2$ with the continuous variables shown first, without loss of generality. From here on for specificity, unless otherwise indicated, the discrete sets $\{0, 1, \ldots, N\}$ and $\{-N_1, \ldots, -1, 0, 1, \ldots, N_2\}$ will be assumed (for $z_1$and z_2 respectively) with N, N_1, N_2 nonnegative integers. First, the complementarity relationship and nonnegativity for z_1, Eq. (1a) can be recast as the following disjunctive constraints (Fortuny-Amat and McCarl 1981):

$$0 \leq q_1 + \begin{pmatrix} A_{11} & A_{12} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \leq M_1 (u) \tag{2a}$$

$$0 \leq z_1 \leq M_1 (1 - u), u_j \in \{0, 1\}, \forall j \tag{2b}$$

where M_1 is a suitably large, positive constant and u is a vector of binary variables. The other constraints (1b) can be used as is and taking Eq. (1b) with Eq. (2) would represent a reformulation of Eq. (1) with just continuous variables z_1, z_2 allowed. If we assume that there were a solution to this version of the original problem, the existence of a solution would not necessarily be guaranteed if we imposed the discrete restrictions from Eqs. (1c) and (1d). The general formulation provided in Gabriel et al. (2012) to solve the discretely-constrained linear, mixed complementarity problem is based on minimizing deviations from complementarity and/or integrality:

$$\min \left[\omega_1 \left[\sum_{r\in D_1} \sum_{i=0}^{N} (\varepsilon_{1ri})^+ + (\varepsilon_{1ri})^- + \sum_{r\in D_2} \sum_{i=-N_1}^{N_2} (\varepsilon_{2ri})^+ + (\varepsilon_{2ri})^- \right] + \omega_2 \left[1^T \sigma \right] \right] \tag{3a}$$

$$0 \leq q_1 + (A_{11}\ \ A_{12}) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \leq M_1 (u) + M_1 \sigma \tag{3b}$$

$$0 \leq z_1 \leq M_1 (1-u) + M_1 \sigma \tag{3c}$$

$$0 = q_2 + (A_{21}\ \ A_{22}) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{3d}$$

$$u_j \in \{0, 1\}, \forall j \tag{3e}$$

$$-M_2 (1 - w_{1ri}) \leq (z_1)_r - i - \varepsilon_{1ri} \leq M_2 (1 - w_{1ri}) \tag{3f}$$

$$i = 0, 1, \ldots, N, r \in D_1$$

$$-M_2 (1 - w_{2ri}) \leq (z_2)_r - i - \varepsilon_{2ri} \leq M_2 (1 - w_{2ri}) \tag{3g}$$

$$i = -N_1, \ldots, -1, 0, 1, \ldots, N_2, r \in D_2$$

$$\varepsilon_{1ri} = (\varepsilon_{1ri})^+ - (\varepsilon_{1ri})^-, i = 0, 1, \ldots, N, r \in D_1 \tag{3h}$$

$$\varepsilon_{2ri} = (\varepsilon_{2ri})^+ - (\varepsilon_{2ri})^-, i = -N_1, \ldots, -1, 0, 1, \ldots, N_2, r \in D_2 \tag{3i}$$

$$\sum_{i=0}^{N} w_{1ri} = 1, \sum_{i=-N_1}^{N_2} w_{2ri} = 1 \tag{3j}$$

$$w_{1ri} \in \{0, 1\}, i = 0, 1, \ldots, N, r \in D_1 \tag{3k}$$

$$w_{2ri} \in \{0, 1\}, i = -N_1, 1, \ldots, N_2, r \in D_2 \tag{3l}$$

$$\sigma \geq 0 \tag{3m}$$

$$(\varepsilon_{1ri})^+, (\varepsilon_{1ri})^- \geq 0, i = 0, 1, \ldots, N, r \in D_1 \tag{3n}$$

$$(\varepsilon_{2ri})^+, (\varepsilon_{2ri})^- \geq 0, i = -N_1, 1, \ldots, N_2, r \in D_2 \tag{3o}$$

Here the variables ε and σ relax integrality and complementarity respectively. The goal of the formulation above is to minimize these deviations. The positive scalar parameters ω_1, ω_2 express the relative importance of the two parts of the objective function: minimizing the relaxations for integrality and complementary, respectively. Choosing their values shows relative importance of relaxing complementarity or integrality. Setting $\omega_1 = 1, \omega_2 = 0$, for example, would mean that deviations from integrality are to be minimized while deviations from complementary can be relaxed. The following two theorems from Gabriel et al. (2012) confirm that a solution to this DC-MLCP exists.

Theorem 1 *Let* $M_2 \geq \max\{N, N_1 + N_2\}$. *Then, this value will be valid for the constraints* (3f) *and* (3g). (*See Gabriel et al.* 2012 *for Proof*)

Assumption 1 Define the set

$$S = \left\{ (z_1, z_2) \,|\, 0 \leq q_1 + \begin{pmatrix} A_{11} & A_{12} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, 0 = q_2 + \begin{pmatrix} A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, z_1 \geq 0 \right\}$$

Then, assume that S is nonempty and there exists a constant M^* such that $M^* \geq \max\{\|z_1\|_\infty, \|z_2\|_\infty\} = \left\| \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\|_\infty$ for all $(z_1, z_2) \in S$.

Infeasibility of the relaxed version of the problem (without the integer restriction) will result in infeasibility of the integer-constrained version as well. Assuming a solution exists for the relaxed problem is automatically guaranteed for certain class of matrices. For example if A_{22} is invertible, then solving for z_2 results in the reduced conditions: $S = \{(z_1) \,|\, 0 \leq (q_1 - A_{12}A_{22}^{-1}q_2) + (A_{11} - A_{12}A_{22}^{-1}A_{21})\, z_1, z_1 \geq 0\}$. By Assumption 1, the LCP$\left((A_{11} - A_{12}A_{22}^{-1}A_{21}),\ (q_1 - A_{12}A_{22}^{-1}q_2)\right)$ needs to be feasible. A sufficient (and stronger condition) is that $(A_{11} - A_{12}A_{22}^{-1}A_{21})$ be an S-matrix (Cottle et al. 1992). Since we assume that the discretely-constrained variables $(z_1)_d \in Z_+, d \in D_1, (z_2)_d \in Z, d \in D_2$ can only take on a finite set of integer values $\{0, 1, \ldots, N\}$ it is not unreasonable that the continuous components $(z_1)_c \in R_+,\ c \in C_1, (z_2)_c \in R,\ c \in C_2$ also be bounded so the second assumption is also reasonable for this setting.

Note that relative to Eqs. (3b) and (3c) it is sufficient to just require that the variables are bounded as stated in the second part of the assumption above. The reason is that if there is an $M^* \geq \max\{\|z_1\|_\infty, \|z_2\|_\infty\}$, then letting $A_{11} = \begin{bmatrix} A_{11}^1 \\ \vdots \\ A_{11}^p \end{bmatrix}$, $A_{12} = \begin{bmatrix} A_{12}^1 \\ \vdots \\ A_{12}^p \end{bmatrix}$ where A_{11}^i, A_{12}^j are respectively, the ith and jth rows of A_{11} and A_{12},

$$q_1 + \begin{pmatrix} A_{11} & A_{12} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \leq q_1 + \begin{pmatrix} \|A_{11}^1\|_1 \|z_1\|_\infty \\ \vdots \\ \|A_{11}^p\|_1 \|z_1\|_\infty \end{pmatrix} + \begin{pmatrix} \|A_{12}^1\|_1 \|z_2\|_\infty \\ \vdots \\ \|A_{12}^p\|_1 \|z_2\|_\infty \end{pmatrix} \leq \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} M_1$$

where

$$M_1 \geq \max\left\{ M^*, \max_i \{(q_1)_i\} + \max_j \left\{ \left\|A_{11}^j\right\|_1 + \left\|A_{12}^j\right\|_1 \right\} M^* \right\} \tag{4}$$

and using the fact that for all $x, y \in R^n$, $|x^T y| \leq \|x\|_1 \|y\|_\infty$ which is a special case of Hölder's inequality (Horn and Johnson 1985). Additionally, if a specific value for M^* is known, then computing M_1 as shown in Eq. (4) is straightforward as it only involves input data in the problem, namely, q_1, A_{11}, and A_{12}.

With this first assumption stated, we have the following theorem.

Theorem 2 *If Assumption* 1 *holds, and* $M_1 \geq M^*$, $M_2 \geq \max\{N, N_1 + N_2\}$ *then problem* (3) *always has a solution.* (*See Gabriel et al.* 2012 *for Proof*)

Note that in this paper, this method requires using the Fortuny-Amat and McCarl reformulation and introducing binary variables. This can result in a large mixed-integer program and can become computationally intensive. However, any method of reformulating complementary constraints as in Siddiqui (2011) and Siddiqui and Gabriel (2012) can be substituted for the Fortuny-Amat and McCarl formulation. Studying the best method for reformulating complementary constraints to use with the method in this paper is the topic of ongoing and future research.

Consider the following DC-Nash game. For instance, there are several Cournot power producers that maximize their profit simultaneously by choosing their optimal production quantities. Their objective function (profit) depends on the production of the competitors through the market demand curve (relationship between the total production and the market price). Players $p = 1, ..., P$ seek optimal values for their decision vectors $\hat{x}^p \in X^p$, $p = 1, ..., P$ by minimizing their cost functions (or negative profit functions) $f^p(\cdot, x^{-p})$ such that

$$f^p\left(\hat{x}^p, \hat{x}^{-p}\right) \leq f^p\left(x^p, \hat{x}^{-p}\right), \forall x^p \in X^p \tag{5}$$

Here $x^p \in R^{n_p}$ represents the variables under player p's control with x^{-p} the remaining variables for the other players. Also, $\hat{x}$ means an equilibrium value to x, and $X^p = C^p \cap Z_+^{n_p}$ where

$$C^p = \left\{ \begin{array}{c} x^p | g_j^p\left(x^p\right) \leq 0,\ j = 1, \ldots, I_p;\ h_k^p\left(x^p\right) = 0 \\ , k = 1, \ldots, E_p;\ x_q^p \geq 0, q \in S_p \end{array} \right\}$$

and $Z_+^{n_p}$ is the set of nonnegative, integer-valued variables, i.e., $x_r^p \in Z_+, r \in \left\{1, \ldots, n_p\right\} \backslash S_p$. Here S_p represents those indices for x^p that relate to continuous variables. A continuous relaxation would then be to replace X^p by C^p, i.e., find $\hat{x}^p,\ p = 1, \ldots, P$ such that

$$f^p\left(\hat{x}^p, \hat{x}^{-p}\right) \leq f^p\left(x^p, \hat{x}^{-p}\right), \forall x^p \in C^p \tag{6}$$

or equivalently that $\hat{x}^p$ solves

$$\min_{x^p} f^p\left(x^p, \hat{x}^{-p}\right) \tag{7a}$$

$$s.t.\ g_j^p\left(x^p\right) \leq 0,\ j = 1, \ldots, I_p \tag{7b}$$

$$h_k^p\left(x^p\right) = 0, k = 1, \ldots, E_p \tag{7c}$$

$$x_q^p \geq 0, q \in S_p \tag{7d}$$

We want the Karush–Kuhn–Tucker (KKT) conditions of Eq. (7) to be equivalent to solving that optimization problem so we assume that the functions $f^p\left(\cdot, x^{-p}\right)$ are convex and a constraint qualification (see Bazaraa et al. 1993, for generalization of these assumptions that will also lead to KKT conditions being sufficient for optimality) holds (e.g., $g_j^p\left(x^p\right), h_k^p\left(x^p\right)$ linear). The KKT conditions for player p's relaxed problem (7) are to find $x^p \in R^{n_p}, \lambda^p \in R^{I_p}, \gamma^p \in R^{E_p}$ such that

$$0 \leq \nabla_{x^p} f^p\left(x^p, x^{-p}\right) + \sum_{j \in I^p} \nabla g_j^p\left(x^p\right) \lambda_j^p + \sum_{k \in E^p} \nabla h_k^p\left(x^p\right) \gamma_k^p \perp x^p \geq 0 \tag{8a}$$

$$0 \leq -g_j^p\left(x^p\right) \perp \lambda_j^p \geq 0,\ j = 1, \ldots, I_p \tag{8b}$$

$$0 = h_k^p\left(x^p\right), \gamma_k^p \text{ free}, k = 1, \ldots, E_p \tag{8c}$$

An interesting question is whether the set of x^p that solves Eq. (8) but with the discrete restrictions for $x_r^p \in Z_+, r \in \left\{1, \ldots, n_p\right\} \backslash S_p$ corresponds to the solution set of the original problem (5). The next result shows that this correspondence is correct.

Theorem 3 *Let S be the set of solutions to the discretely-constrained Nash game Eq.* (5) *and T be the set of solutions to Eq.* (8) *for which $x_r^p \in Z_+, r \in \left\{1, \ldots, n_p\right\} \backslash S_p$. Then, S=T.*

Proof Let $\hat{x}^p \in T$, then $\hat{x}^p$ solves

$$\min_{x^p} f^p\left(x^p, \hat{x}^{-p}\right) \tag{9a}$$

$$s.t.\ g_j^p\left(x^p\right) \leq 0,\ j = 1, \ldots, I_p \tag{9b}$$

$$h_k^p\left(x^p\right) = 0, k = 1, \ldots, E_p \tag{9c}$$

$$x_q^p \geq 0, q \in S_p \tag{9d}$$

$$x_r^p \in Z_+, r \in \left\{1, \ldots, n_p\right\} \backslash S_p \tag{9e}$$

or that

$$f^p\left(\hat{x}^p, \hat{x}^{-p}\right) \leq f^p\left(x^p, \hat{x}^{-p}\right),$$

$$\forall x^p \in C^p \cap \left\{x_r^p \in Z_+ | r \in \left\{1, \ldots, n_p\right\} \backslash S_p\right\}$$

$$\Leftrightarrow f^p\left(\hat{x}^p, \hat{x}^{-p}\right) \leq f^p\left(x^p, \hat{x}^{-p}\right), \forall x^p \in X^p$$

so that $\hat{x}^p \in S$. Clearly the steps are reversible so the result is shown.

To be able to end up with a linear, mixed-integer program, we restrict the payoff function to be quadratic and the constraint functions to be linear, that is

$$f^p\left(x^p, x^{-p}\right) = \frac{1}{2}\begin{pmatrix} x^p \\ x^{-p} \end{pmatrix}^T \begin{pmatrix} N_1^p & N_2^p \\ N_2^p & N_3^p \end{pmatrix}\begin{pmatrix} x^p \\ x^{-p} \end{pmatrix} + \left(c^p\right)^T x^p \tag{10}$$

and

$$g_j^p\left(x^p\right) = \left(d_j^p\right)^T x^p - \kappa^p \leq 0,\ j = 1, \ldots, I_p \tag{11}$$

$$h_k^p\left(x^p\right) = \left(e_k^p\right)^T x^p - \delta^p = 0,\ k = 1, \ldots, E_p \tag{12}$$

To reformulate the continuous relaxation of the original problem (5), we use the complementarity problem form of the Nash problem suitably relaxed as shown in Eq. (8). These KKT conditions are equivalent to a set of disjunctive constraints of the form:

$$0 \leq \nabla_{x^p} f^p\left(x^p, x^{-p}\right) + \sum_{j \in I^p} \nabla g_j^p\left(x^p\right)\lambda_j^p + \sum_{k \in E^p} \nabla h_k^p\left(x^p\right)\gamma_k^p \leq M_1^p u^p \tag{13a}$$

$$0 \leq x^p \leq M_1^p\left(1 - u^p\right) \tag{13b}$$

$$0 \leq -g_j^p\left(x^p\right) \leq M_1^p v_j^p,\ j = 1, \ldots, I_p \tag{13c}$$

$$0 \leq \lambda_j^p \leq M_1^p\left(1 - v_j^p\right),\ j = 1, \ldots, I_p \tag{13d}$$

$$0 = h_k^p\left(x^p\right),\ \gamma_k^p \text{ free},\ k = 1, \ldots, E_p \tag{13e}$$

$$u^p \in \{0, 1\}^{n_p} \tag{13f}$$

$$v^p \in \{0, 1\}^{I_p} \tag{13g}$$

for a suitably large value of M_1^p that can be computed as decribed in Theorems 1 and 2. An alternative method is also provided in Gabriel and Leuthold (2010). Using the quadratic form of f^p and the linear forms of g^p and h^p from above, results in the following linear, mixed-integer program with arbitrary objective function $\sum_{p=1}^{P} (z^p)^T x^p$ and the integer restrictions added back:

$$\min_{x^p} \sum_{p=1}^{P} \left(z^p\right)^T x^p \tag{14a}$$

$$s.t. \text{ for all } p = 1, \dots, P$$

$$0 \le \frac{1}{2}\left(N_1^p + N_1^{pT}\right) x^p + \frac{1}{2}\left(N_2^p + N_2^{pT}\right) x^{-p} + c^p$$
$$+ \sum_{j \in I^p} d_j^p \lambda_j^p + \sum_{k \in E^p} e_k^p \gamma_k^p \le M_1^p u^p \tag{14b}$$

$$0 \le x^p \le M_1^p \left(1 - u^p\right) \tag{14c}$$

$$0 \le -\left(d_j^p\right)^T x^p + \kappa^p \le M_1^p v_j^p, \ j = 1, \dots, I_p \tag{14d}$$

$$0 \le \lambda_j^p \le M_1^p \left(1 - v_j^p\right), \ j = 1, \dots, I_p \tag{14e}$$

$$0 = \left(e_k^p\right)^T x^p - \delta^p, \gamma_k^p \text{ free}, k = 1, \dots, E_p \tag{14f}$$

$$u^p \in \{0, 1\}^{n_p}, v^p \in \{0, 1\}^{I_p} \tag{14g}$$

$$x_r^p \in Z_+, r \in \left\{1, \dots, n_p\right\} \setminus S_p \tag{14h}$$

We only need feasibility of this problem and the arbitrary objective function is just included as a potential lever and for purposes of moving onto the next formulation. Of course the above problem may be infeasible for several reasons. First, the original problem (5) may itself not be feasible due to incompatible constraints in the players' problems. Second, the problem may not be feasible since there may be a conflict between integrality of the vector $x_r^p \in Z_+, r \in \left\{1, \dots, n_p\right\} \setminus S_p$ and complementarity of the system enforced after the fact via the disjunctive constraints.

To ensure that the above reformulation does not have a conflict between complementarity and integrality,[1] the following relaxed version of the problem is employed.

$$\min \omega_1 \left[\sum_{p=1}^{P} \sum_{r \in \{1,\dots,n_p\} \setminus S_p} \sum_{i=0}^{N} \left(\varepsilon_{ri}^p\right)^+ + \left(\varepsilon_{ri}^p\right)^- \right] + \omega_2 \left[1^T \left(\sigma^p + \psi^p\right)\right] \tag{15a}$$

$$s.t. \text{ for all } p = 1, \dots, P$$

$$0 \le \frac{1}{2}\left(N_1^p + N_1^{pT}\right) x^p + \frac{1}{2}\left(N_2^p + N_2^{pT}\right) x^{-p}$$

[1] We assume that the continuous form of the problem is feasible.

$$+ c^p + \sum_{j \in I^p} d_j^p \lambda_j^p + \sum_{k \in E^p} e_k^p \gamma_k^p \leq M_1^p \left(u^p\right) + M_1^p \sigma^p \tag{15b}$$

$$0 \leq x^p \leq M_1^p \left(1 - u^p\right) + M_1^p \sigma^p \tag{15c}$$

$$0 \leq -\left(d_j^p\right)^T x^p + \kappa^p \leq M_2^p \left(v_j^p\right) + M_2^p \psi_j^p \tag{15d}$$

$$0 \leq \lambda_j^p \leq M_1^p \left(1 - v_j^p\right) + M_1^p \psi_j^p,\ j = 1, \ldots, I_p \tag{15e}$$

$$0 = \left(e_k^p\right)^T x^p - \delta^p,\ \gamma_k^p \text{ free},\ k = 1, \ldots, E_p \tag{15f}$$

$$u^p \in \{0, 1\}^{n_p} \tag{15g}$$

$$v^p \in \{0, 1\}^{I_p} \tag{15h}$$

$$-M_1^p \left(1 - w_{ri}^p\right) \leq x_r^p - i - \varepsilon_{ri}^p \leq M_1^p \left(1 - w_{ri}^p\right), \tag{15i}$$

$$i = 0, 1, \ldots, N, r \in \left\{1, \ldots, n_p\right\} \backslash S_p \tag{15j}$$

$$\varepsilon_{ri}^p = \left(\varepsilon_{ri}^p\right)^+ - \left(\varepsilon_{ri}^p\right)^- \tag{15k}$$

$$\sum_{i=0}^{N} w_{ri}^p = 1 \tag{15l}$$

$$w_{ri}^p \in \{0, 1\},\ i = 0, 1, \ldots, N, r \in \left\{1, \ldots, n_p\right\} \backslash S_p \tag{15m}$$

$$\sigma^p, \psi^p \geq 0 \tag{15n}$$

$$\left(\varepsilon_{ri}^p\right)^+, \left(\varepsilon_{ri}^p\right)^- \geq 0, \forall r, i \tag{15o}$$

In the above formulation (15), the ε_{ri}^p are used to target the specified integer values and σ^p, ψ^p are used to relax complementarity, both of which are minimized in the objective function weighting the two objective function parts with positive weights ω_1 and ω_2. Thus, minimizing these deviations helps find the optimal integer solution, as described in Gabriel et al. (2012).

3 Numerical examples

This section presents the results of numerical examples for solving discretely-constrained Nash–Cournot games from the theory outlined in Section 2. The first example constrains the production quantities to be integer while the second example has continuous production quantities but binary startup/shutdown variables. In both examples, seven variations are considered. These variations go through different relaxation techniques and combinations of formulations to be described later. The problems selected can be shown to have unique solutions by simple algebra.

The results show that formulation (15) provides solutions to the original discretely-constrained problems. The variations also show that, as stated before, Eq. (14) can lead to an infeasible solution. Moreover, relaxing complementarity in Eq. (14) but keeping integer restrictions also leads to a discrete feasible solution. Both numerical examples show that relaxing complementarity is essential to obtaining discrete solutions. Enforcing discrete restrictions, even by integer relaxation, does not help obtain the integer solutions and relaxation of complementary conditions is necessary. A combination of both, as presented in Eq. (15) helps obtain the required solutions in both cases. A relaxation of integrality has a somewhat easy interpretation and in fact is commonly used in equilibrium problems with discrete restrictions. The relaxation in complementarity is a bit more novel. One interpretation is that the resulting equilibrium values (e.g., quantities, prices) are those values that are the minimum distance to the solution set of the relaxed problem for which integrality is maintained. From that perspective, these equilibrium values can be construed as a projection of the relaxed solution on to the discretely-constrained feasible region.

3.1 Problem definition

For ease of presentation and comparison but with no loss of generality, consider a Nash–Cournot game with two players ($p = 1, 2$). Given an inverse demand curve $Price = a - b\,(Quantity)$, each player chooses $q_p \in \mathbb{Z}_+$ to maximize their profit function

$$Profit_p = Price \times q_p - (\beta_p q_p^2 + \rho_p q_p) \tag{16}$$

where the term in parentheses denotes cost as a function of quantity selected i.e., q_p. The formulation of the game is the same as discussed above.

For the first example, let, $a = 6, b = 1, \beta_1 = \beta_2 = 1,$ and $\rho_1 = \rho_2 = 1,$ as well as adding capacity constraints for both players of the form

$$q_p \leq q_{\max} \tag{17}$$

where $q_{\max} = 4$. Since only integer-valued production q_p is allowed, a bimatrix payoff table (assuming maximizing payoff) as shown below in Table 1 is employed to solve Eq. (5).

Table 1 Bimatrix Nash–Cournot game, profits (q_1/q_2)

	0	1	2	3	4
0	(0, 0)	(0, 3)	(0, 2)	(0, −3)	(0, −12)
1	(3, 0)	(2, 2)	(1, 0)	(0, −6)	(−1, −16)
2	(2, 0)	(0, 1)	(−2, −2)	(−4, −9)	(−6, −20)
3	(−3, 0)	(−6, 0)	(−9, −4)	(−12, −12)	(−15, −24)
4	(−12, 0)	(−16, −1)	(−20, −6)	(−24, −15)	(−28, −28)

Table 2 Bimatrix Nash–Cournot game, profits (q_1/q_2)

(only adjustments: $a = 9$, $\rho_2 = 3$)					
	0	1	2	3	4
0	(0, 0)	(0, 4)	(0, 4)	(0, 0)	(0, −8)
1	(6, 0)	(5, 3)	(4, 2)	(3, −3)	(2, −12)
2	(8, 0)	(6, 2)	(4, 0)	(2, −6)	(0, −16)
3	(6, 0)	(3, 1)	(0, −2)	(−3, −9)	(−6, −20)
4	(0, 0)	(−4, 0)	(−8, −4)	(−12, −12)	(−16, −24)

Clearly, $q_1 = 1$, $q_2 = 1$ is the unique Nash equilibrium in pure strategies. Another way to solve Nash–Cournot games is by simultaneously solving the problems

$$\max_{q_p} \left[a - b\ (q_1 + q_2)\right] q_p - \left(\beta_p q_p^2 + \rho_p q_p\right)$$

$$s.t.\ q_p \le q_{\max} \quad (\lambda_p\ dual)$$

$$q_p \ge 0$$

for $p = 1, 2$. Since the slope of the inverse demand function $b > 0$ and $\beta_p > 0$, the KKT conditions are both necessary and sufficient for solving these problems. These conditions are to find $q_1, q_2, \lambda_1, \lambda_2$ that solve the following LCP:

$$0 \le 2q_p(b + \beta_p) + bq_{-p} - (a - \rho_p) + \lambda_p \perp q_p \ge 0 \tag{18a}$$

$$0 \le q_{\max} - q_p \perp \lambda_p \ge 0 \tag{18b}$$

for each $p = 1, 2$. However, the KKT conditions are only valid if $q_p, p = 1, 2$ are continuous-valued. Thus, the resulting LCP needs to avoid discrete restrictions on the q_p variables. In this particular example, solving the above LCP after assuming $q_p \in \mathbb{R}_+$ results in the integer solution $q_1 = 1, q_2 = 1$ with $Price = 4$.

However, changing some of the data to $a = 9$ and $\rho_2 = 3$ results in a non-integer solution of $q_1 = 1.733$, $q_2 = 1.067$, and $Price = 6.2$. But the new bimatrix payoff table for the original discrete version of this game with these new data (Table 2), shown below gives a unique discrete solution of $q_1 = 2$, $q_2 = 1$ with $Price = 6$.

This example shows what can happen if the relaxed LCP does not provide integer-valued answers. In the next section, more numerical tests are described with the new data $a = 9$, $b = 1$, $\beta_1 = \beta_2 = 1$,$\rho_1 = 1$, and $\rho_2 = 3$.

3.2 Relaxing integrality and complementarity

In this section, several variations on relaxing complementarity and/or integrality are numerically explored.

The first variation is to solve the continuous version of the LCP (i.e., without any integer restrictions) relating to Eq. (5) ("MLCP") . Solving the original version of the problem with the integer restrictions relating to Eq. 5 is variation

Table 3 Description of formulation variations

Variation	$\sigma-$compl.	$\varepsilon-$integr.	Problem desc.
1	No	No	MLCP
2	No	No	Bimatrix
3	No	No	Integer variables
4	Yes	No	Integer variables
5	Yes	No	Cont. variables
6	No	Yes	Cont. variables
7	Yes	Yes	Cont. variables

2 ("Bimatrix") and is solved by examining the bimatrix payoff table. In the remaining variations to be described, there are two ways of forcing integrality of the solutions. First, the problem can be integer-constrained by indicating to the solver that the variables can only take on integer values (variations 3 and 4) with variation 4 also relaxing complementarity and variation 3 enforcing exact complementarity. Second, in variation 5, complementarity can be relaxed without constraining the problem to have integer solutions, hence "continuous variables" for the problem description. Hence, we should not expect integer solutions. Finally, in variations 6 and 7, integers can be targeted using the ε deviational variables (15). In variation 6, no relaxation for complementarity is allowed. Variation 7 allows relaxation for both complementarity and integrality. Table 3 describes the various possible formulations considered. Note that Variation 5 is equivalent to setting $\omega_1 = 0, \omega_2 = 1$ and Variation 6 is equivalent to setting $\omega_1 = 1, \omega_2 = 0$. For Variation 7, we set $\omega_1 = 0.5, \omega_2 = 0.5$. Other combinations of values of ω_1, ω_2 were tested but not shown here as they provided the same solution as either Variations 5, 6 or 7 for all numerical examples. One can think of this as a tradeoff between complementary and integrality, commonly used in multiobjective optimization. Note that the values of M_1, and M_2 were set equal to 1000, which is a larger value than required by the discussion in the previous subsection.

Tables 4 and 5 summarize the results.

Table 4 shows that a solution to the integer-constrained Nash game is to have $q_1 = 2, q_2 = 1$ with a resulting price of 6 (variation 2). When the integer restrictions are removed, the solution is then $q_1 = 1.733, q_2 = 1.067$ with the new price of 6.2 (variation 1). Solving the MIP version of the problem but forcing exact complementarity and integrality results in an infeasible solution (variation 3) as would be expected. Interestingly, the original integer solution

Table 4 Summary of results

$(a = 9, b = 1, \beta_1 = \beta_2 = 1, \rho_1 = 1, \rho_2 = 3)$			
Var.	Solution (q_1, q_2)	Price	Profits (P1, P2)
1	(1.733, 1.067)	6.2	(6.01, 2.28)
2	(2, 1)	6	(6, 2)
3	Infeasible	Infeasible	Infeasible
4	(2, 1)	6	(6, 2)
5	(1.733, 1.067)	6.2	(6.01, 2.28)
6	(1.733, 1.067)	6.2	(6.01, 2.28)
7	(2, 1)	6	(6, 2)

Table 5 Summary of results

$(a = 9, b = 1, \beta_1 = \beta_2 = 1, \rho_1 = 1, \rho_2 = 3)$		
Variation	Sum ϵ	Sum σ
1	N/A	N/A
2	N/A	N/A
3	N/A	N/A
4	N/A	0.002
5	N/A	0
6	0.334	N/A
7	0	0.002

to the Nash problem can be obtained with the MIP approach as long as complementarity is relaxed (variation 4) or when integers are targeted using ε's (without enforcing integrality) along with the complementarity relaxation (variation 7). It is interesting to note that variation 7 is numerical validation to obtain integer solutions to the DC-Nash game. From the perspective of accuracy in attaining the original production values and price, the MIP approach is correct in this instance and thus provides an alternative, viable method for solving such problems. It is interesting to note the difference in results between variations 4 and 5. The former achieves the correct integer solution but directly forces the variables in GAMS to be integer-valued. The latter allows relaxation of complementarity but does not give integer solutions as expected. Furthermore, variation 6 also does not get the correct integer solution even though the using the ε deviational variables were included.

3.3 Example relevant to production systems

In many applications, the quantities q_p are actually positive real numbers but there are also constraints of the form

$$s_p q_{\min} \leq q_p \leq s_p q_{\max} \tag{19}$$

where s_p is a binary variable that is 1 when the player p chooses to produce and 0 when player p chooses to not produce. Here the binary variable s_p might for example relate to the on/off status for a power generation unit. If on, then the minimum and maximum production quantities are in force. If off, then both the upper and lower bounds are equal to zero. The original capacity constraint is replaced by the one above and the resulting Nash–Cournot game is then solved with $a = 9, b = 1, \beta_1 = \beta_2 = 1, \rho_1 = 1, \rho_2 = 3, q_{\min} = 1.5$, and $q_{\max} = 4$. The binary variables s_p are the ones targeted when complementarity and integrality are relaxed but still allowing for continuous generation variables. The following tables summarize the results (Tables 6 and 7), with q_1, q_2 always continous variables.

The solutions to this example are very different from the previous one.

Variation 2 shows the true solution when the variables s_p, $p = 1, 2$ are forced to be binary. Namely, player 2 produces at the minimum level of 1.5 but player 1 chooses a value of 1.625, in between the minimum and maximum. The continuous relaxation (variation 1) achieves higher profits for both players as would be expected due to less restrictive constraints but does not end up

Table 6 Summary of results

Example relevant to production systems			
Var.	Solution (q_1, q_2)	Binary(s_1, s_2)	Profits (P1, P2)
1	(1.733, 1.067)	(0.347, 0.213)	(6.01, 2.28)
2	(1.625, 1.5)	(1, 1)	(5.28, 2.06)
3	(1.625, 1.5)	(1, 1)	(5.28, 2.06)
4	(1.625, 1.5)	(1, 1)	(5.28, 2.06)
5	(1.733, 1.067)	(0.347, 0.711)	(6.01, 2.28)
6	(1.625, 1.5)	(1, 1)	(5.28, 2.06)
7	(1.625, 1.5)	(1, 1)	(5.28, 2.06)

with binary values for the s_p variables. Interestingly, all other variations on relaxation are able to achieve the correct production quantities (q_p) and binary production indicators (s_p) except for variation 5 when only complementarity is relaxed. For this particular problem, forcing integrality is key through the solver in variations 3 and 4 or by minimizing ε as in variations 6 and 7, as all give the correct binary solution for s_p, $p = 1, 2$.

3.4 Example of a power network

As a third example, consider a power market with two producers supplying to one demand node as shown in Fig. 1. Producers 1 and 2 choose to produce quantitites q_1 and q_2 respectively, and supply it to meet inelastic demand d, while there are transmission lines (with flow variables q_{12}, q_{13}, q_{23}) between the three nodes. There is a marginal utility of demand c_d and marginal costs c_1 and c_2 for producers 1 and 2, respectively. There is also a market operator who maximizes its own profits by buying from the producers and selling to the consumers.

The producer p $(p = 1, 2)$ solves the following optimization problem

$$\min_{q_p} \{c_p q_p - \lambda_n q_p\} \tag{20}$$

$$0 \le q_p \le q_p^{\max} \quad (\beta_p^{\max}) \tag{21}$$

where λ_n is the (endogenous) price at node n. Note that the producer p is active at node $n = p$.

The market operator solves the following optimization problem (with $\overline{q_p}$ introduced to have a square system). The equality constraints set the power

Table 7 Summary of results

Var.	Price	Sum ϵ	Sum σ
1	6.2	N/A	N/A
2	5.875	N/A	N/A
3	5.875	N/A	N/A
4	5.875	N/A	0
5	6.2	N/A	0
6	5.875	0	N/A
7	5.875	0	0

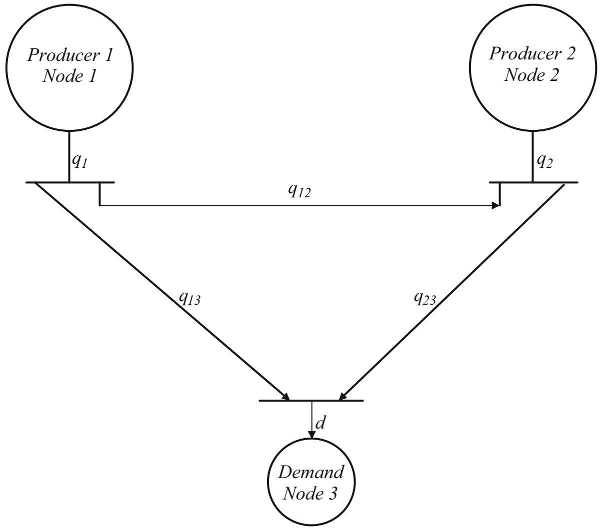

Fig. 1 Power network with two producers

flow (q_{13} for example, signifies flow from node 1 to node 3) equal to the power produced and the inequality constraints give a bound on the maximum amount of flow allowed. Flow can be towards the opposite direction as well which is signified by a negative number (i.e., if q_{13} is negative, then the flow is from node 3 to node 1), so the inequalities contain a maximum negative flow as well. Note that for simplicity, the conditions that relate power flows to voltages are not included.

$$\min_{\overline{q_1},\overline{q_2},d,q_{13},q_{23},q_{12}} \{c_1\overline{q_1} + c_2\overline{q_2} - c_d d\} \tag{22}$$

$$q_{13} + q_{12} - \overline{q_1} = 0 \quad (\lambda_1) \tag{23}$$

$$q_{23} - q_{12} - \overline{q_2} = 0 \quad (\lambda_2) \tag{24}$$

$$d - q_{13} - q_{23} = 0 \quad (\lambda_3) \tag{25}$$

$$-q_{12}^{\max} \leq q_{12} \leq q_{12}^{\max} \quad (\beta_{12}^{\min}, \beta_{12}^{\max}) \tag{26}$$

$$-q_{13}^{\max} \leq q_{13} \leq q_{13}^{\max} \quad (\beta_{13}^{\min}, \beta_{13}^{\max}) \tag{27}$$

$$-q_{23}^{\max} \leq q_{23} \leq q_{23}^{\max} \quad (\beta_{23}^{\min}, \beta_{23}^{\max}) \tag{28}$$

Table 8 Parameter values used in example

$q_1^{\max}$	$q_2^{\max}$	$q_{12}^{\max}$	$q_{13}^{\max}$	$q_{23}^{\max}$	c_1	c_2	c_d
18	20.5	12	15	15	2	1	5

Table 9 Description of formulation variations

Variation	σ−compl.	ε−integr.	Problem Desc.
1	No	No	MLCP
3	No	No	Integer variables
4	Yes	No	Integer variables
5	Yes	No	Cont. variables
6	No	Yes	Cont. variables
7	Yes	Yes	Cont. variables

Additional balancing constraints, which are included this problem are below.

$$\overline{q_1} = q_1 \tag{29}$$

$$\overline{q_2} = q_2 \tag{30}$$

The above optimization problems can be combined to form an MCP, which gives a solution to the game. Our goal here is to see if we restricted the quantities produced and flows to be integer-valued, if we can come up with an equilibrium solution. The following Table 8 gives the values of the parameters used for solving this network problem.

Hence, producer 2 has a lower marginal cost so will attempt to supply more units of q_2. We use the same process as before and formulate the problem according the variations in Table 9. Note that we are not considering the bimatrix game for this example, so there is no variation 2. The full formulation of variation 7 is similar to the formulation of the previous Section 3.3, variation 7 model, found in the Appendix. The value of $M = \max\{M_1, M_2\}$ chosen here was 100. Sensitivity tests were performed to check this value, and it was seen that any value over $M = 31$ worked well. Table 10 shows the results for the example under different variations.

Note that again, variation 7 gives an integer solution. Comparison to variation 4 is critical, as both of them give the same solution. However, variation 7 provides integer solutions but does not explicitly enforce integrality, while

Table 10 Solution to power market example

Variations	1	3	4	5	6	7
q_1	9.5	Infeasible	10	9.5	9.5	10
q_2	20.5	Infeasible	20	20.5	20.5	20
q_{12}	−5.5	Infeasible	−5	−5.5	−5.5	−5
q_{13}	15	Infeasible	15	15	15	15
q_{23}	15	Infeasible	15	15	15	15
λ_1	2	Infeasible	2	2	2	2
λ_2	2	Infeasible	2	2	2	2
λ_3	5	Infeasible	5	5	5	5
d	30	Infeasible	30	30	30	30
Sum ε	N/A	N/A	N/A	N/A	1	0
Sum σ	N/A	N/A	0.5	0	N/A	0.5

variation 4 requires imposing integer restrictions to get to the answer. Variation 3 proves to be infeasible, while variations 5 and 6 show that only including σ-complementarity or only including ε-integrality is not sufficient to achieve an integer solution for all the variables that are constrained as such. Note that prices at each node ($\lambda_1, \lambda_2, \lambda_3$) stay the same at each node, regardless of the variation. However, variation 3 did not provide any solution, so not only does variation 7 provide an integer solution, it does so without imposing integer restrictions and also delivering reasonable prices.

4 Conclusions

This paper proposes a methodology to solve discretely-constrained Nash games formulated as mixed complementarity problems. This has so far been a mathematical exploration into the idea of relaxing complementary to solve discretely-constrained Nash games. While economic interpretation is beyond the scope of this paper, it is part of ongoing research. As pointed out by an anonymous referee, the deviations from complementary can be interpreted as an economic measure. The discrete restrictions can lead to infeasible solutions, so a relaxation is needed. However, we have shown that relaxing only integer restrictions does not necessarily yield an integer solution. This paper provides a complementarity relaxation as well. From the theoretical analysis carried out and the examples considered, the following conclusions can be drawn:

1. Relaxing both integrality and complementarity enables the selection of an integer, equilibrium solution.
2. The relaxed problem formulated in Eq. (15) allows analyzing the tradeoff between complementarity and integrality. This is done by actually computing the cost of integrality in terms of complementarity and, conversely, the cost of complementarity in terms of integrality.
3. Three examples are used to illustrate the technique proposed and its practical relevance to power markets.

Appendix

A.1 Variation 7 formulation

Variation 7 for the example in Section 3.3 where both complementarity and integrality are relaxed is shown below, where all variables unless specified otherwise are taken to be nonnegative.

$$\min \left\{ \sum_p \sum_i (\epsilon_{pi})^+ + (\epsilon_{pi})^- + \sum_p \sum_j (\sigma_{jp} + \tau_{jp}) \right\} \tag{A1}$$

$$0 \leq 2q_1(b + \beta_1) + bq_2 - (a - \rho_1) + \lambda_1 - \eta_1 \leq M_{11}u_{11} + M_{11}\sigma_{11}$$

$$0 \leq 2q_2(b + \beta_2) + bq_1 - (a - \rho_2) + \lambda_2 - \eta_2 \leq M_{12}u_{12} + M_{12}\sigma_{12}$$
$$0 \leq -\lambda_1 q_{\max} + \eta_1 q_{\min} + \gamma_1 \leq M_{31}u_{31} + M_{31}\sigma_{31}$$
$$0 \leq -\lambda_2 q_{\max} + \eta_2 q_{\min} + \gamma_2 \leq M_{32}u_{32} + M_{32}\sigma_{32}$$
$$0 \leq q_1 \leq M_{11}(1 - u_{11}) + M_{11}\sigma_{11}$$
$$0 \leq q_2 \leq M_{12}(1 - u_{12}) + M_{12}\sigma_{12}$$
$$0 \leq c_1 \leq M_{31}(1 - u_{31}) + M_{31}\sigma_{31}$$
$$0 \leq c_2 \leq M_{32}(1 - u_{32}) + M_{32}\sigma_{32}$$
$$0 \leq -q_1 + c_1 q_{\max} \leq M_{21}v_{21} + M_{21}\tau_{21}$$
$$0 \leq -q_2 + c_2 q_{\max} \leq M_{22}v_{22} + M_{22}\tau_{22}$$
$$0 \leq q_1 - c_1 q_{\min} \leq M_{41}v_{41} + M_{41}\tau_{41}$$
$$0 \leq q_2 - c_2 q_{\min} \leq M_{42}v_{42} + M_{42}\tau_{42}$$
$$0 \leq -c_1 + 1 \leq M_{61}v_{61} + M_{61}\tau_{61}$$
$$0 \leq -c_2 + 1 \leq M_{62}v_{62} + M_{62}\tau_{62}$$
$$0 \leq \lambda_1 \leq M_{21}(1 - v_{21}) + M_{21}\tau_{21}$$
$$0 \leq \lambda_2 \leq M_{22}(1 - v_{22}) + M_{22}\tau_{22}$$
$$0 \leq \eta_1 \leq M_{41}(1 - v_{41}) + M_{41}\tau_{41}$$
$$0 \leq \eta_2 \leq M_{42}(1 - v_{42}) + M_{42}\tau_{42}$$
$$0 \leq \gamma_1 \leq M_{61}(1 - v_{61}) + M_{61}\tau_{61}$$
$$0 \leq \gamma_2 \leq M_{62}(1 - v_{62}) + M_{62}\tau_{62}$$
$$u_{jp} \in \{0, 1\}, v_{jp} \in \{0, 1\} \qquad p = 1, 2$$
$$-M(1 - w_{pi}) \leq c_p - i - \epsilon_{pi} \leq M(1 - w_{pi}),$$
$$\epsilon_{pi} = (\epsilon_{pi})^+ - (\epsilon_{pi})^-$$
$$\sum_i w_{pi} = 1,\ p = 1, 2;\ w_{pi} \in \{0, 1\}\, i = 0, 1$$

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